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Some problems of BCk, BCI algebras

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In my seminar on algebras and logics, we took up two subjects since 1960's. One of them was to be algebraic formulations of set-theoretic operations. On this problem, since Dedekind, we have an abstract formulation on union and intersection. They are included in the lattice theory. But there is no theory directly based on the set difference. Can we generalize the set difference operation in an algebraic form. We am interested in analytic and projective sets. (1) Can we formulate Hausdorff, Kolmogorov, Kantrovic-Livenson operations and related ones in a pure algebraic form(see [7],[8],[13])?

It concerns with algebras with operations of infinite type([14]).

From the viewpoint of the theory of categories, there is a problem: (2) find a universal property which characterize the set difference in the category SET of sets.

On the other hand, from the viewpoint of logic, Lukasiewicz-Lesniewski style systems of logic were useful to formulate our new algebras. Consequently we recognized the set difference and Lukasiewicz C theory are both sides, and we formulated BCK, BCI-algebras(1966) from Meredith systems. Until now, both algebras have been rapidly developed by many workers(see [9]), and it grew up a large theory.[1]

Let X be a set with a binary operation $*$ and a constant 0. Then the system $(X, *, 0)$ is a BCK if it satisfies the following axioms:

(1) $((x * y) * (x * z)) * (z * y) = 0$,

(2) $(x * (x * y)) * y = 0$,

(3) $x * x = 0$,

(4) $0 * x = 0$,

(5) $x * y = 0 \wedge y * x = 0 \Rightarrow x = y$.

If X is finite, the system is called finite. The order is the cardinality of X.

Example 1. Let X be the set N of natural numbers $0,1,\ldots,n,\ldots$. For any two natural numbers m, n we define

$$m * n = \begin{cases} m - n & \text{if } n < m, \\ 0 & \text{if } m \leq n. \end{cases}$$

Then $(N, *, 0)$ is a a linear BCK. Under the same condition, the left of any cut of N makes a finite BCK. It follows from this example that there exists a BCK of order $n(= 2,3,\ldots)$.

An interesting fact is that a BCK of order n always contains a sub BCK of order $n - 1$. We have very important problem which has not

[1] Mathematics Subject classification 2000 includes 06F35 BCK, BCI-algebras as one of items. Both algebras are mentioned in Encyclopedia of Math.(Supplement volume) of Kluwer Academic Publishers.

yet completely solved: (3) Construct an algorithm to find all BCKs of order n(n = 2,3,...). The complete tables of BCK-algebras of order 2 3, 4 and 5 are mentioned in Meng-Jun [9].

order	2	3	4	5
number	1	3	14	88

I asked my colleagues to establish a computer program of BCK-algebras of all finite orders many years ago(In those days, personal computers were not popular). (4) Find the automorphism group of each BCK [12].

Give a BCK X of order n. Then (5) make all BCKs of order n + 1 containing X as a sub BCK.

The class of BCK is a very wide class in which there are many important subclasses. The original class of BCK does not form a variety, as shown(1983) by A.Wronski and Y.Komori independently. Considering this fact, I would like to omit "algebra" following our recent customs, so I call them briefly BCK or BCI.

Some elementary results on BCK are stated as follows:

(a) Any BCK is a partially ordered set under an order $x \leq y$ which is defined by "$x \leq y \Leftrightarrow x * y = 0$".

(b) $x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x$.

(c) $(x * y) * z = (X * z) * y$. (permutation rule)

If there exists an element 1 of BCK such that $x \leq 1$ for all $x \in X$, then X is said to be bounded.

$1 * x$ is denoted by Nx. Then we have

$N1 = 0$, $N0 = 1$. $NNx \leq x$, $NNNx = Nx$. $Nx * y = Ny * x$.

If x satisfies $NNx = x$ then x is called an involution. The set of involutions in a bounded BCK is a subalgebra and

$$x * Ny = y * Nx$$

holds.

We consider some subclasses of BCK. In set theory $A - (A - B)$ for any sets A, B represents the intersection of A and B.

S.Tanaka considered the condition

$$(6) \qquad x * (x * y) = y * (y * x),$$

and the class of BCK with (6) is said to be commutative. The class of commutative BCKs is a variety. An axiom system is given by H.Yutani (1977) as follows:

(6) $x * (x * y) = y * (y * x)$,

(7) $(x * y) * z = (x * z) * y$,

(3) $x * X = 0$,

(8) $x * 0 = x$.

As well known, there are two main algebras, namely positive implicative algebra and implicative algebra. These algebras are also defined in the frame of BCK as follows.

BCK X is said to be positive implicative if it satisfies

$$(x * z) * (y * z) = (x * y) * z$$

for all $x, y, z \in X$.

BCK X is said to be implicative if for any $x, y \in X$

$$x = x * (y * x)$$

holds in X.

Both classes make varieties. X is an implicative BCK if and only if X is commutative and positive implicative. Moreover, a bounded implicative BCK is a Boolean algebra.

In our research, we introduce two new classes of BCK. One of them is a class in which contains commutative BCK and all finite BCK. This class is called a quasi-commutative BCK which was introduced by H. Yutani(1977).

For x, y $\in$ X, we inductively define

$$Q_{0,0}(x, y) = x * (x * y),$$

$$Q_{m+1,n} = Q_{m,n}(x, y) * (x * y),$$

$$Q_{m,n+1} = Q_{m,n}(x, y) * (y * x).$$

A commutative BCK is defined as $Q_{0,0}(x, y) = Q_{0,0}(y, x)$.

A BCK is called quasi-commutative of type (i,j,m,n) if

$$Q_{i,j}(x, y) = Q_{m,n}(x, y).$$

All finite algebras are quasi-commutative(H.Yutani).

X is positive implicative $\Leftrightarrow$ $Q_{0,1}(x, y) = Q_{0,1}(y, x)$, and X is implicative $\Leftrightarrow$ $Q_{1,0}(x, y) = Q_{0,0}(y, x)$ or $Q_{0,1}(x, y) = Q_{0,0}(y, x)$.

A quasi-commutative of type (i,j,m,n) is equationally definable (H.Yutani).

There are several important unsolved problems. One of them is (6) to find a proper quasi-commutative BCK of type (i,j,m,n), where i,j,m, and n are non-negative integers. Except some special cases, we have not any informations about this.

Another class is a BCK with condition (S).

If there exists the greatest element x satisfying $x * a \leq b$ for any a, b, then it is called a BCK with condition (S). The greatest element is denoted by a o b.

Then any BCK X with condition (S) is a commutative semigroup with respect to o, and 0 is a zeroelement of X. In such a BCK, $(x * y) * z = x * (y \circ z)$ and $x \leq y \Rightarrow x \circ z \leq y \circ z$ for $x, y, z \in X$ holds. If X is bounded, then $1 \circ x = 1$ holds.

For a BCK X with condition (S), the following propositions are equivalent:

1) X is positive implicative,

2) $x \leq y \Rightarrow x \circ y = y$,

3) $x \circ x = x$ for every $x \in X$,

4) $(x \circ y) * z = (x * z) \circ (y * z)$.

A finite BCK with condition (S) has at least two idemptents, namely the element having the property $x \circ x = x$(Any compact semi-group has at least one idemptent).

In a BCK with condition (S), it is known that various ideals introduced in a BCK coincide. The positive implicativity is also characterized by the behavior of such ideals.

Recently, W.A.Dudek[1] gave an algebraic formulation of Lesniewski classical (Equivalential) Calculus([5],[6]) following our ideas.

An axiom system of this calculus([4],[5]) is given by

(*) Epp, EEpqEEqrErp.

There are several single axiom systems which were discovered by Lukasiewicz, namely

EEpqEErpEqr, EEPqEEprErq, EEpqEErqEpr.

Moreover, there are some others:

EEEpqrEqErp, EEEpEqrqErp.

We do not know (7) whether two expressions of (*) are independent or not.

There are some unsolved probelms related to variable functors by Lesniewski. For example, Boolean algebra is defined as

$$C\delta\ \delta 0 \delta p$$

with a variable functor(= variable logical connectives) δ.

(8) Can we formulate an axiom system of BCK, BCI with variable functors?

On the other hand, we can consider a theory with variable operations, so called the hyperidentity theory of BCK(for detail, see [10], [11]).

Since Q.P.Hu research [2], several mathematicians have tried the generalizations of the concepts of BCK, BCI.

(X, *, 0) is saide to be BCH if it satisfies

$$x * x = 0,$$

$$(x * y) * z = (x * z) * y,$$

$$x * y = y * x = 0 \Rightarrow x = y.$$

On 1994, A.Ursini (and P.Agliano) defines more general algebra, namely, a subtrative algebra. This algebra has a binary operation s and a constant 0. Axioms are

$$s(x, x) = 0, \qquad (x * x = 0),$$

$$s(x, 0) = x. \qquad (x * 0 = x).$$

It follows from the definition that the class of subtractive algebras forms a variety.

On such generalizations, they will be mentioned in my writing "A way to BCK and related systems" published in Math. Japonica vol.52(2000).

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